Switching Algebra: Principle of Duality

- **Any theorem or identity in switching algebra remains true if 0 and 1 are swapped and . and + are swapped.**
- **True because all the duals of all the axioms are true, so duals of all switching-algebra theorems can be proven using the duals of axioms.**
- **Dual of a logic expression: For a fully parenthesized logic expression $F(X_1, X_2, \ldots, X_n, +, \cdot, ')$ then the dual expression, F^D, is the same expression with +, . swapped:**

$$F^D(X_1, X_2, \ldots, X_n, +, \cdot, ') = F(X_1, X_2, \ldots, X_n, \cdot, +, ')$$

- **The generalized DeMorgan's theorem T14 can be stated as:**

$$[F(X_1, X_2, \ldots, X_n)]' = F^D(X_1', X_2', \ldots, X_n')$$

Switching Algebra Axioms & Theorems

(A1)	$X = 0$ if $X \neq 1$	(A1')	$X = 1$ if $X \neq 0$	
(A2)	If $X = 0$, then $X' = 1$	(A2')	if $X = 1$, then, $X' = 0$	
(A3)	$0 . 0 = 0$	(A3')	$1 + 1 = 1$	
(A4)	$1 . 1 = 1$	(A4')	$0 + 0 = 0$	
(A5)	$0 . 1 = 1 . 0 = 0$	(A5')	$1 + 0 = 0 + 1 = 1$	
(T1)	$X + 0 = X$	(T1')	$X . 1 = X$	(Identities)
(T2)	$X + 1 = 1$	(T2')	$X . 0 = 0$	(Null elements)
(T3)	$X + X = X$	(T3')	$X . X = X$	(Idempotency)
(T4)	$(X')' = X$ (Involution)			
(T5)	$X + X' = 1$	(T5')	$X . X' = 0$	(Complements)
(T6)	$X + Y = Y + X$	(T6')	$X . Y = Y . X$	(Commutativity)
(T7)	$(X + Y) + Z = X + (Y + Z)$	(T7')	$(X . Y) . Z = X . (Y . Z)$	(Associativity)
(T8)	$X . Y + X . Z = X . (Y + Z)$	(T8')	$(X + Y) . (X + Z) = X + Y . Z$	(Distributivity)
(T9)	$X + X . Y = X$	(T9')	$X . (X + Y) = X$	(Covering)
(T10)	$X . Y + X . Y' = X$	(T10')	$(X + Y) . (X + Y') = X$	(Combining)
(T11)	$X . Y + X'. Z + Y . Z = X . Y + X' . Z$			
(T11')	$(X + Y) . (X' + Z) . (Y + Z) = (X + Y) . (X' + Z)$			(Consensus)
(T12)	$X + X + \ldots + X = X$	(T12')	$X . X . \ldots . X = X$	(Generalized idempotency)
(T13)	$(X_1 . X_2 . \ldots . X_n)' = X_1' + X_2' + \ldots + X_n'$			
(T13')	$(X_1 + X_2 + \ldots + X_n)' = X_1' . X_2' . \ldots . X_n'$			(DeMorgan's theorems)
(T14)	$[F(X_1, X_2, \ldots, X_n, +, .)]' = F(X_1', X_2', \ldots, X_n', ., +)$			(Generalized DeMorgran's theorem)

Logic Expression Algebraic Manipulation Example

- **Prove that the following identity is true using Algebraic expression Manipulation : (one can also prove it using a truth table)**

$$X . Y + X . Z = ((X' + Y') . (X' + Z'))'$$

  - **Starting from the left hand side of the identity:**

    Let $F = X . Y + X . Z$

    $A = X . Y \qquad B = X . Z$

    Then $F = A + B$

  - **Using DeMorgan's theorem T 13 on F:**

    $F = A + B = (A' . B')' \qquad (1)$

  - **Using DeMorgan's theorem T 13' on A, B:**

    $A = X . Y = (X' + Y')' \qquad (2)$

    $B = X . Z = (X' + Z')' \qquad (3)$

  - **Substituting A, B from (2), (3), back in F in (1) gives:**

    $F = (A' . B')' = ((X' + Y') . (X' + Z'))'$

  Which is equal to the right hand side of the identity.

Standard Representations of Logic Functions

- **Truth table for n-variable logic function:**

  Input combinations are arranged in 2^n rows in ascending binary order, and the output values are written in a column next to the rows.
- **Practical for functions with a small number of variables.**
- **The general structure of a 3-variable truth table is given by:**

Row	X	Y	Z	F(X,Y,Z)
0	0	0	0	F(0,0,0)
1	0	0	1	F(0,0,1)
2	0	1	0	F(0,1,0)
3	0	1	1	F(0,1,1)
4	1	0	0	F(1,0,0)
5	1	0	1	F(1,0,1)
6	1	1	0	F(1,1,0)
7	1	1	1	F(1,1,1)

Truth table for a specific function:

Row	X	Y	Z	F
0	0	0	0	1
1	0	0	1	0
2	0	1	0	0
3	0	1	1	1
4	1	0	0	1
5	1	0	1	0
6	1	1	0	1
7	1	1	1	1

Logic Function Representation Definitions

- ***A literal:*** **is a variable or a complement of a variable**

  Examples: X, Y, X', Y'

- ***A product term:*** **is a single literal, or a product of two or more literals.**

  Examples: Z' W.Y.Y X.Y'.Z W'.Y'.Z

- ***A sum-of-products expression:*** **is a logical sum of product terms.**

  Example: Z' + W.X.Y + X.Y'.Z + W'.Y'.Z

- ***A sum term:*** **is a single literal or logical sum of two or more literals**

  Examples: Z' W + X + Y X + Y' + Z W' + Y' + Z

- ***A product-of-sums expression:*** **is a logical product of sum terms.**

  Example: Z'. (W + X + Y) . (X + Y' + Z) . (W' + Y' + Z)

- ***A normal term:*** **is a product or sum term in which no variable appears more than once**

  Examples of non-normal terms: W.X.X.Y' W+W+X'+Y X.X'.Y

  Examples of normal terms: W . X . Y' W + X' + Y

Logic Function Representation Definitions

- **Minterm**

  An n-variable minterm is a normal product term with n literals.

  There are 2^n such products terms.

  Example of 4-variable minterms:

  W.X'.Y'.Z'   W.X.Y'.Z   W'.X'.Y.Z'

- **Maxterm**

  An n-variable maxterm is a normal sum term with n literals.

  There are 2^n such sum terms.

  Examples of 4-variable maxterms:

  W' + X' + Y + Z'   W + X' + Y' + Z   W' + X' + Y + Z

- **A minterm can be defined as as product term that is 1 in exactly one row of the truth table.**

- **A maxterm can similarly be defined as a sum term that is 0 in exactly one row in the truth table.**

Minterms/Maxterms for A 3-variable function F(X,Y,Z)

Row	X	Y	Z	F	Minterm	Maxterm
0	0	0	0	F(0,0,0)	X'.Y'.Z'	X + Y + Z
1	0	0	1	F(0,0,1)	X'.Y'.Z	X + Y + Z'
2	0	1	0	F(0,1,0)	X'.Y.Z'	X + Y' + Z
3	0	1	1	F(0,1,1)	X'.Y.Z	X + Y' + Z'
4	1	0	0	F(1,0,0)	X.Y'.Z'	X' + Y + Z
5	1	0	1	F(1,0,1)	X.Y'.Z	X' + Y + Z'
6	1	1	0	F(1,1,0)	X.Y.Z'	X' + Y' + Z
7	1	1	1	F(1,1,1)	X.Y.Z	X' + Y' + Z'

Canonical Sum Representation

- **Minterm number:**

  minterm i refers to the minterm corresponding to row i of the truth table. For n-variables i is in the set

  $$\{0,1, \ldots, 2^n-1\}$$

- **The canonical sum representation of a logic function is a sum of the minterms corresponding to the truth table rows for which the function produces a 1 output.**
- **A short-hand representation of the minterm list uses the S notation and minterm numbers to indicate the sum of minterms of the function.**
- **This representation is usually realized using 2-level AND-OR logic circuits with inverters at AND gates inputs as needed.**

Canonical Sum Example

- **The function represented by the truth table:**

Row	X	Y	Z	F
0	0	0	0	1
1	0	0	1	0
2	0	1	0	0
3	0	1	1	1
4	1	0	0	1
5	1	0	1	0
6	1	1	0	1
7	1	1	1	1

has the canonical sum representation:

$$F = S_{X,Y,Z}(0, 3, 4, 6, 7)$$

Minterm list using Σ notation

$$= X'.Y'.Z' + X'.Y.Z + X.Y'.Z' + X.Y'.Z' + X.Y.Z$$

Algebraic canonical sum of minterms

Canonical Product Representation

- Maxterm i refers to the maxterm corresponding to row i of the truth table. For n-variables i is in the set

$$\{0,1, \ldots, 2^n-1\}$$

- The canonical product representation of a logic function is the product of the maxterms corresponding to the truth table rows for which the function produces a 0 output.
- The product of such minterms is called a maxterm list
- A short-hand representation of the maxterm list uses the P notation and maxterm numbers to indicate the product of maxterms of the function.
- This representation is usually realized using 2-level OR-AND logic circuits with inverters at OR gates inputs as needed.

Canonical Product Example

- **The function represented by the truth table:**

Row	X	Y	Z	F
0	0	0	0	1
1	0	0	1	0
2	0	1	0	0
3	0	1	1	1
4	1	0	0	1
5	1	0	1	0
6	1	1	0	1
7	1	1	1	1

has the canonical product representation:

$$F = P_{X,Y,Z}(1,2,5)$$ ← **Maxterm list using Π notation**

$$= (X + Y + Z') \cdot (X + Y' + Z) \cdot (X' + Y + Z')$$

Algebraic canonical product of maxterms

Conversion Between Minterm/Maxterm Lists

- **To convert between a minterm list and a maxterm list take the set complement.**

Examples:

$$S_{X,Y,Z}(0,1,2,3) = P_{X,Y,Z}(4,5,6,7)$$

$$S_{X,Y}(1) = P_{X,Y}(0,2,3)$$

$$S_{W,X,Y,Z}(0,1,2,3,5,7,11,13) = P_{W,X,Y,Z}(4,6,8,9,12,14,15)$$